Interplay between Graph Isomorphism and Earth Mover's Distance in the Query and Communication Worlds

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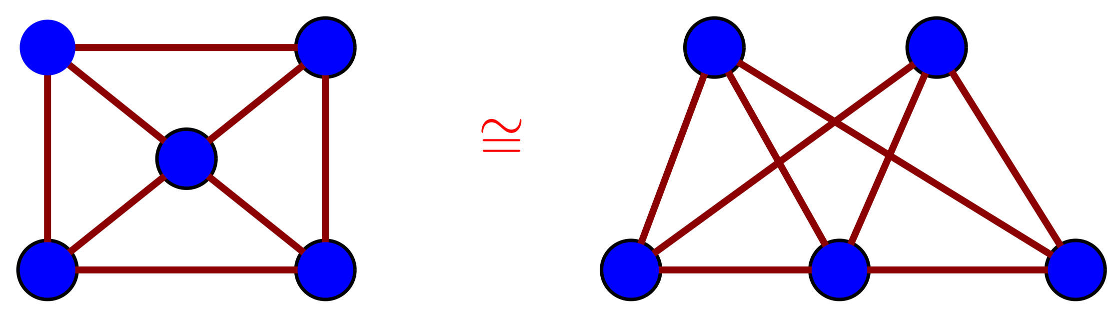
Tolerant Testing of Graph Isomorphism

Graph Isomorphism

Graphs G_u and G_k are isomorphic if there exists a bijection $\psi : V(G_u) \to V(G_k)$ such that for all pair of vertices $u, v \in V(G_u)$, the edges $\{u, v\} \in E(G_u)$ if and only if $\{\psi(u), \psi(v)\} \in E(G_k)$.



GI Distance

For a bijection $\phi : V_u \to V_k$, the distance between G_u and G_k is defined as

$$d_\phi(G_u, G_k) := \left| \left\{ (u, v) : \begin{array}{l} \text{Exactly one among } (u,v) \in E_u \\ \text{or } (\phi(u), \phi(v)) \in E_k \text{ holds} \end{array} \right\} \right|$$

The GI Distance between G_u and G_k is defined as

$$d(G_u, G_k) := \min_{\phi : V_u \to V_k} d_\phi(G_u, G_k)$$

How to access a graph?

- Graphs can be accessed by querying the entries of the adjacency matrix.

(ϵ_1, ϵ_2)-GI Testing

For a known graph G_k and an unknown graph G_u (both on n vertices) and two proximity parameters ϵ_1 and ϵ_2 with $0 \le \epsilon_1 < \epsilon_2 \le 1$, want to decide whether $d(G_u, G_k) \le \epsilon_1 n^2$ or $d(G_u, G_k) \ge \epsilon_2 n^2$?

What is the query complexity of (ϵ_1, ϵ_2)-GI testing?

Known Result

Fischer-Matsliah (SICOMP'08) proved that the query complexity of non-tolerant GI (when $\epsilon_1 = 0$), is $\widetilde{\Theta}(\sqrt{n})$.

Tolerant Testing of EMD

Earth Mover Distance (EMD)

The *Earth Mover's Distance* between two probability distributions p and q over a Hamming cube $H = \{0,1\}^n$ is denoted by $EMD(p_u, p_k)$ and defined as the optimum solution to the following linear program:

$$\textbf{min} \sum_{i,j \in H} f_{ij} d_H(i,j)$$

$$\textbf{Subject to} \sum_{j \in H} f_{ij} = p_u(i) \; \forall i \in H \;\; \& \;\; \sum_{i \in H} f_{ij} = p_k(j) \; \forall j \in H$$

How to access a distribution?

- Accessing a distribution via samples is equivalent to samples **with** replacement from a multi-set. Thus

$$EMD(S_u, S_k) \triangleq n \cdot EMD(p_u, p_k)$$

- For multi-sets, both sampling **with** and **without** replacement are possible. We focus on sampling **without** replacement model here.

(γ_1, γ_2)-EMD Testing

For a known probability distribution p_k and an unknown probability distribution p_u, accessed by samples **without** replacement, want to decide whether $EMD(p_u, p_k) \le \gamma_1$ or $EMD(p_u, p_k) \ge \gamma_2$ for $0 \le \gamma_1 < \gamma_2 \le 1$?

What is the minimum number of samples WITHOUT replacement required for (γ_1, γ_2)-EMD testing?

Known Results

- $O(n)$ samples **without** replacement are enough to decide (γ_1, γ_2)-EMD. This follows from learning unknown distribution
- The best lower bound of (γ_1, γ_2)-EMD is $\Omega(n^{1-o(1)})$ samples **without** replacement following Valiant (STOC'08).

Theorem: Tolerant GI Testing ≡ Tolerant EMD Testing WITHOUT replacement

Let $Q_{GI}(G_u, G_k)$ denote the number of adjacency queries to G_u, required to decide (ϵ_1, ϵ_2)-GI and $QWOR_{EMD}$ denote the number of samples **without** replacement required to decide (γ_1, γ_2)-EMD. Then

$$Q_{GI}(G_u, G_k) = \widetilde{\Theta}\left(QWOR_{EMD}(n)\right)$$

Sketch of Proof

- **Upper bound**: Non trivial generalization of Fischer-Matsliah's algorithm of non-tolerant testing.
- **Lower bound**: Follows from a *pure* reduction.

Implications

- The query complexity for tolerant GI testing is $\Omega(n^{1-o(1)})$.
- Lower bound reduction technique holds for other computation models, say for communication complexity model.
- Using our lower bound reduction, we give a proof of Fischer-Matsliah's lower bound for $(0, \epsilon_2)$-GI.

Communication Complexity

- **Question:** Alice and Bob have graphs G_A and G_B respectively and want to decide if $d(G_A, G_B)$ is less than ϵ_1 or more than ϵ_2 by communicating among themselves.
- Randomized communication complexity of non tolerant GI is $O(1)$ whereas deterministic communication complexity of non tolerant GI (hence tolerant GI) is $\Omega(n^2)$
- Our theorem gives an equivalence of tolerant GI and tolerant EMD in the communication setting.

Open Problems

- Can Valiant-Valiant's lower bound (STOC'11) of $\Omega(\frac{n}{\log n})$ for (γ_1, γ_2)-EMD testing **with** replacement be extended to **without** replacement?
- What is the randomized communication complexity of (ϵ_1, ϵ_2)-GI?